Journal of Engineering Mathematics **32:** 237–256, 1997.

High-order approximation equations for the primitive equations of the atmosphere

T. TACHIM MEDJO[1], R. TEMAM[1,2] and S. WANG[1]
[1] *The Institute for Scientific Computing and Applied Mathematics, Indiana University, Bloomington, IN 47405, U.S.A.*
[2] *Laboratoire d'Analyse Numerique, Université Paris-Sud, Batiment 425, 91405 Orsay, France*

Received 21 November 1996; accepted in revised form 20 February 1997

Abstract. In this article, a family of models approximating the primitive equations of the atmosphere, which are known to be the fundamental equations of the atmosphere, is presented. The primitive equations of the atmosphere are used as a starting point and asymptotic expansions with respect to the Rossby number are considered to derive the nth-order approximate equations of the primitive equations of the atmosphere. Simple global models of the atmosphere are obtained. These higher-order models are linear and of the same form (with different right-hand sides, depending on the lower-order approximations) as the (first-order) global quasi-geostrophic equations derived in an earlier article. From a computational point of view, there are two advantages. Firstly, all the models are linear, so that they are easy to implement. Secondly, all order models are of the same form, so that, with slight modifications, the numerical code for the (first-order) global quasi-geostrophic model can be employed for all higher-order models. From a physical point of view, higher-order models capture more physical phenomena, such as the meridional flows, even though they are small in magnitude. Of course, there are still many subtle issues involved in this project, such as the convergence of the asymptotics; they will be addressed elsewhere. The article is concluded by a presentation of numerical simulations based on these models.

Key words: atmosphere, Rossby number, numerical simulation, computation, geostrophic flow.

Introduction

The general equations describing the motion and state of the atmosphere are very complicated. To solve these equations numerically is still one of the great challenges in weather forecasting. Simplified models are usually introduced for numerical computations [1–4].

There are two essential characteristics of the atmosphere which are used in simplifying these equations. The first one is that, for large-scale geostrophysical flows, the ratio between the vertical and the horizontal scales is very small; this leads to the primitive equations for the atmosphere [5–15].

Another small parameter is the Rossby number $\varepsilon = Ro$, which is the ratio of the speed of (horizontal) wind to the speed of rotation of the earth around the polar axis. For the atmosphere this number is of order 1/50.

In [10] the authors considered asymptotic expansions of the primitive equations of the atmosphere with respect to the Rossby number and derived a very simple global-circulation model of the atmosphere, the global quasi-geostrophic model (GQG), for which the equations of motion for the wind and the temperature are linear evolution equations similar to the linear Stokes equations. In particular, the following results were proved:

- the zeroth order motions are independent of the longitude, wind traveling exactly towards east or west, and
- the first-order equations are linear.

In comparison with the classical mid-latitude (mesoscale) quasi-geostrophic theory ([1–3, 11–13]), the Coriolis parameter is an order-1 function, and the β-plane assumption is no longer used (see [11]). It should be noted that we *do not* expand the Coriolis parameter in terms of the Rossby number ε, and that we *retain* the spherical geometry of the earth.

Compared to some important parameters in the primitive equations of the atmosphere such as the viscosity, which may be very small for some realistic flows, the Rossby number is still a relatively large constant. Moreover, even though the meridional component of the wind is relatively small by comparison with the zonal component, the meridional flow is also extremely important in numerical weather prediction.

The objectives of this article are twofold. Firstly, we want to validate numerically the GQG model derived in [10] by purely mathematical arguments (asymptotic expansions); we address this question in Section 4. Then we want to derive and study the higher-order approximations of the global geostrophic asymptotics, leading to higher-order models, and retaining the meridional motion of the general circulation.

The new idea here is to decompose *properly* the solution (v^n, T^n) of the nth-order approximation; we write:

$$v^n = \widetilde{v}^n + \omega^n, \qquad T^n = \widetilde{T}^n + T_*^n.$$

Here (ω^n, T_*^n) is determined by lower-order approximations, and $(\widetilde{v}^n, \widetilde{T}^n)$ satisfies the same equations (with different right-hand sides, involving lower-order approximations) as the (first-order) global geostrophic equations. Although the derivation procedure is somewhat involved, the final equations for solving $(\widetilde{v}^n, \widetilde{T}^n)$ and calculating (ω^n, T_*^n) are, surprisingly, very natural and simple. Some advantages of these higher-order models are as follows:

(a) the equations are linear and of the same form as the first-order equations. Therefore, they are relatively easy to implement. With little extra work on the right-hand sides that involve only lower-order approximations, the same numerical code as for the (first-order) GQG applies to all the higher-order approximations. Of course, special attention is still needed to handle the difficulties at the poles (see [16]);

(b) the higher-order models capture the meridional motion in a simple way. This is important from a numerical-weather-prediction point of view.

As in [10], we present our global geostrophic expansion for the PEs on the whole globe. However, we would like to emphasize that, as in all geostrophic theories, the global geostrophic model introduced in [10] and in the present article has limitations in the tropical region, owing to the degeneracy of the Coriolis parameter at the equator.

The article is organized as follows: In the first section, we recall from [10] the primitive equations of the atmosphere and the global quasi-geostrophic equations of the atmosphere. In Section 2 we study the second-order approximate equations in the asymptotic expansions with respect to the Rossby number. Section 3 generalizes the previous method to the nth-order approximate equations, for any integer number n. Section 4 is devoted to the presentation of some numerical simulations that use these models.

1. The primitive equations (PEs)

In this section, we briefly recall from [10] the primitive equations of the atmosphere and the global quasi-geostrophic equations.

We start with the formulation of the nondimensional PEs which we obtain by integrating the diagnostic equations in the pressure direction. In this context the equations read (see [10] for details):

$$\left.\begin{aligned}
&\varepsilon\left[\frac{\partial v}{\partial t}+\nabla_v v-W(v)\frac{\partial v}{\partial\eta}\right]+fk\times v+\nabla\Phi_s+\nabla\mathcal{M}(T/K_2)+\varepsilon L_1 v=0,\\
&\varepsilon\alpha\left[\frac{\partial}{\partial t}+\nabla_v-W(v)\frac{\partial}{\partial\eta}\right]T-\frac{W(v)}{K_2}+\varepsilon L_2 T=\varepsilon Q,\\
&\operatorname{div}\int_0^1 v\,\mathrm{d}\eta=0.
\end{aligned}\right\}\tag{1.1}$$

The initial and boundary value conditions are:

$$\left.\begin{aligned}
&\frac{\partial v}{\partial\eta}=\gamma_s(v-v_s),\qquad \frac{\partial T}{\partial\eta}=\alpha_s(T-T_s),\quad \text{for }\eta=0,\\
&\frac{\partial v}{\partial\eta}=0,\quad \frac{\partial T}{\partial\eta}=0,\quad \text{for }\eta=1;\quad u=(v,T)=u_0=(v_0,T_0),\quad \text{at }t=0,
\end{aligned}\right\}\tag{1.2}$$

where $u_0=(v_0,T_0)$ is a given function (initial data).

The notations used above are as follows (see [10] for more details):

(1) The nondimensional pseudo-spatial domain is given by

$$M=S^2\times(0,1),$$

with coordinate system (θ,φ,η). Here, θ is the colatitude $(0\leqslant\theta\leqslant\pi)$, φ is the longitude $(0\leqslant\varphi\leqslant 2\pi)$, and η is the nondimensional pressure

$$\eta=(P-p)/(P-p_0),$$

$0<p_0<P$, representing the top of the atmosphere and the surface of the earth.

(2) The unknown functions are the 2D horizontal velocity v, the temperature T, and the geopotential Φ_s on the surface of the earth $(\eta=0)$, *i.e.* the value of $\Phi=gz$ at the isobar $p=P$ located above the surface of the earth. The vertical velocity $\omega=W(v)$ is given by

$$W(v)=-\operatorname{div}\mathcal{M}^*v.$$

(3) The linear operators L_1 and L_2, representing the dissipation, are given by

$$L_1=-\frac{1}{Re_1}\Delta-\frac{1}{Re_2}\frac{\partial}{\partial\eta}\left(K_1\frac{\partial}{\partial\eta}\right),\qquad L_2=-\frac{1}{Rt_1}\Delta-\frac{1}{Rt_2}\frac{\partial}{\partial\eta}\left(K_1\frac{\partial}{\partial\eta}\right).$$

We use Δ,∇, div to denote the 2D horizontal (in θ and φ directions) Laplacian, gradient, and divergence operators. The averaging operators $\mathcal{M}$ and $\mathcal{M}^*$ are given by

$$\mathcal{M}\Psi=\int_0^\eta\Psi\,\mathrm{d}\eta',\qquad \mathcal{M}^*\Psi=\int_\eta^1\Psi\,\mathrm{d}\eta'.$$

(4) The parameters $Re_1,Re_2,Rt_1,Rt_2,\gamma_s$ and α_s are positive constants, $K_1=K_1(\eta)$ is a smooth positive function, T_s and v_s are given functions. The nondimensional parameter ε is the Rossby number defined by

$$\varepsilon=\frac{V}{2\Omega a},$$

V being the typical horizontal velocity of the wind, Ω the angular velocity of the earth, and a the radius of the earth.

We now recall the global geostrophic asymptotics introduced in [10]. The basic idea behind the global geostrophic asymptotics is that, for the planetary-scale atmosphere (the horizontal scale is of order of a, the radius of the earth), the Coriolis parameter f is an order-1 function. It has to be treated as a variable function, and the β-plane in classical geostrophic asymptotics is no longer valid. Hence, as we mentioned in the introduction, we do not expand f in terms of the Rossby number ε.

We proceed as follows (see [10] for more details) and set formally

$$\left.\begin{array}{ll} v = v^0 + \varepsilon v^1 + \varepsilon^2 v^2 + \cdots, & T = T^0 + \varepsilon T^1 + \varepsilon^2 T^2 + \cdots, \\ \Phi = \Phi^0 + \varepsilon \Phi^1 + \varepsilon^2 \Phi^2 + \cdots, & \Phi_s = \Phi_s^0 + \varepsilon \Phi_s^1 + \varepsilon^2 \Phi_s^2 + \cdots. \end{array}\right\} \tag{1.3}$$

Substituting formally (1.3) in (1.1)–(1.2), we obtain (see [10]) the following approximate equations of the PEs:

(a) *Zero-order approximation*. At the zeroth order, $O(1)$, we have (see [10])

$$\left.\begin{array}{ll} fk \times v^0 + \nabla \Phi^0 = 0, & \operatorname{div} \displaystyle\int_0^1 v^0 \, \mathrm{d}\eta = 0, \\ \operatorname{div} \displaystyle\int_0^1 v^0 \, \mathrm{d}\eta' = 0, & \Phi^0 = \Phi_s^0 + \mathcal{M}(T^0/K_2), \end{array}\right\} \tag{1.4}$$

which gives

$$fk \times v^0 + \nabla \Phi^0 = 0, \qquad \operatorname{div} v^0 = 0, \qquad T^0 = K_2 \frac{\partial \Phi^0}{\partial \eta}. \tag{1.5}$$

(b) *First-order approximation*. At the level $O(\varepsilon)$ we have:

$$\left.\begin{array}{l} \dfrac{\partial v^0}{\partial t} + \nabla_{v^0} v^0 + fk \times v^1 + \nabla \Phi^1 + L_1 v^0 = 0, \\ \alpha \left[\dfrac{\partial}{\partial t} + \nabla_{v^0}\right] T^0 + \dfrac{\operatorname{div} \mathcal{M}^* v^1}{K_2} + L_2 T^0 = Q, \\ \operatorname{div} \displaystyle\int_0^1 v^1 \, \mathrm{d}\eta = 0, \qquad T^1 = K_2 \frac{\partial \Phi^1}{\partial \eta}, \qquad \Phi^1 = \Phi_S^1 + \mathcal{M}(T^1/K_2). \end{array}\right\} \tag{1.6}$$

Regrouping (1.5) and (1.6) as in [10], we find the following global quasi-geostrophic Equations (1.7)–(1.9):

$$\left.\begin{array}{l} \dfrac{\partial v^0}{\partial t} + \nabla_{v^0} v^0 + fk \times v^1 + \nabla \Phi^1 + L_1 v^0 = 0, \\ \alpha \left[\dfrac{\partial}{\partial t} + \nabla_{v^0}\right] T^0 + \dfrac{\operatorname{div} \mathcal{M}^* v^1}{K_2} + L_2 T^0 = Q, \end{array}\right\} \tag{1.7}$$

$$\operatorname{div} v^0 = 0, \qquad fk \times v^0 + \nabla \Phi^0 = 0, \qquad T^0 = K_2 \frac{\partial \Phi}{\partial \eta}, \tag{1.8}$$

$$\operatorname{div} \int_0^1 v^1 \, \mathrm{d}\eta = 0, \qquad T^1 = K_2 \frac{\partial \Phi^1}{\partial \eta}, \tag{1.9}$$

with the initial and boundary value conditions:

$$\left.\begin{aligned}&\frac{\partial v^0}{\partial \eta}=\gamma_s(v^0-v_s^0),\qquad \frac{\partial T^0}{\partial \eta}=\alpha_s(T^0-T_s^0)\quad \text{at}\ \eta=0,\\ &\frac{\partial v^0}{\partial \eta}=0,\qquad \frac{\partial T^0}{\partial \eta}=0\quad \text{at}\ \eta=1;\qquad (v^0,T^0)|_{t=0}=(v_0^0,T_0^0).\end{aligned}\right\}\tag{1.10}$$

It is proven in [10] that v_θ^0 vanishes identically and that v_φ^0 and T^0 are independent of the longitude φ. Consequently, the global quasi-geostrophic equations can be rewritten as follows:

$$\frac{\partial v_\varphi^0}{\partial t}+\mathcal{L}_1 v_\varphi^0=-f\overline{v}_\theta^1,\qquad \alpha\frac{\partial T^0}{\partial t}+\mathcal{L}_2 T^0=\overline{Q}-\frac{1}{K_2}\int_\eta^1\frac{1}{\sin\theta}\frac{\partial(\overline{v}_\theta^1\sin\theta)}{\partial\theta}\,\mathrm{d}\eta',\tag{1.11}$$

$$v^0=v_\varphi^0 e_\varphi=\frac{1}{f}\frac{\partial\Phi^0}{\partial\theta}e_\varphi,\qquad T^0=K_2\frac{\partial\Phi^0}{\partial\eta},\qquad \Phi^0\ \text{is independent of}\ \varphi,\tag{1.12}$$

$$\int_0^1\frac{1}{\sin\theta}\frac{\partial(\overline{v}_\theta^1\sin\theta)}{\partial\theta}\,\mathrm{d}\eta'=0.\tag{1.13}$$

The boundary and initial conditions are deduced from (1.10) by integration in φ:

$$\left.\begin{aligned}&\frac{\partial v^0}{\partial \eta}=\gamma_s(v^0-\overline{v}_s^0),\qquad \frac{\partial T^0}{\partial \eta}=\alpha_s(T^0-\overline{T}_s^0),\quad \text{at}\ \eta=0,\\ &\frac{\partial v^0}{\partial \eta}=0,\qquad \frac{\partial T^0}{\partial \eta}=0,\quad \text{at}\ \eta=1;\qquad (v^0,T^0)=(v_0^0,T_0^0),\quad \text{at}\ t=0,\end{aligned}\right\}\tag{1.14}$$

(v_0^0,T_0^0) being a function independent of φ which we have to prescribe (to choose), based on an asymptotic expansion similar to (1.3), of the initial data v_0,T_0 in (1.2). In the above equations,

$$\overline{Q}=\overline{Q}(\theta,\eta,t)=\frac{1}{2\pi}\int_0^{2\pi}Q(\theta,\varphi,\eta,t)\,\mathrm{d}\varphi,\tag{1.15}$$

is a given function. The operators $\mathcal{L}_1$ and $\mathcal{L}_2$ are defined by

$$\left.\begin{aligned}&\mathcal{L}_1 v_\varphi^0=-\frac{1}{Re_1}\left(\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial v_\varphi^0}{\partial\theta}\right)-\frac{v_\varphi^0}{\sin^2\theta}\right)-\frac{1}{Re_2}\frac{\partial}{\partial\eta}\left(K_1\frac{\partial v_\varphi^0}{\partial\eta}\right),\\ &\mathcal{L}_2 T^0=-\frac{1}{Rt_1\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial T^0}{\partial\theta}\right)-\frac{1}{Rt_2}\frac{\partial}{\partial\eta}\left(K_1\frac{\partial T^0}{\partial\eta}\right).\end{aligned}\right\}\tag{1.16}$$

In (1.11)–(1.14), $\overline{v}_\theta^1$ is the Lagrange multiplier of the constraint (1.12). The existence and uniqueness of solutions of the global quasi-geostrophic Equations (1.11)–(1.14) is studied in detail in [10].

2. Second-order approximate equations

Based on the methodology used in [10], we now derive the second-order approximate equations. Throughout this section, we suppose that the solution (v_φ^0, T^0) of the global quasi-geostrophic Equations (1.11)–(1.14) is sufficiently regular. More generally, all functions are assumed to be sufficiently regular.

At the level $O(\varepsilon^2)$ we have (thanks to the zeroth and first-order results):

$$\left.\begin{aligned} &\left[\frac{\partial v^1}{\partial t} + \nabla_{v^1} v^0 + \nabla_{v^0} v^1 - W(v^1)\frac{\partial v^0}{\partial \eta} + fk \times v^2 + \nabla \Phi^2 + L_1 v^1\right] = 0, \\ &\alpha\left[\frac{\partial T^1}{\partial t} + \nabla_{v^0} T^1 + \nabla_{v^1} T^0 - W(v^1)\frac{\partial T^0}{\partial \eta}\right] - \frac{W(v^2)}{K_2} + L_2 T^1 = 0, \end{aligned}\right\} \tag{2.1}$$

$$\operatorname{div} \int_0^1 v^2 \,\mathrm{d}\eta = 0, \qquad T^2 = K_2 \frac{\partial \Phi^2}{\partial \eta}, \qquad \Phi^2 = \Phi_S^2 + \mathcal{M}(T^2/K_2), \tag{2.2}$$

with the initial and boundary conditions

$$\left.\begin{aligned} &\frac{\partial v^1}{\partial \eta} = \gamma_s(v^1 - v_s^1), \qquad \frac{\partial T^1}{\partial \eta} = \alpha_s(T^1 - T_s^1), \quad \text{at } \eta = 0, \\ &\frac{\partial v^1}{\partial \eta} = 0, \qquad \frac{\partial T^1}{\partial \eta} = 0, \quad \text{at } \eta = 1; \qquad (v^1, T^1)|_{t=0} = (v_0^1, T_0^1). \end{aligned}\right\} \tag{2.3}$$

Hereafter, we derive a simplified form of the second-order approximate Equations (2.1)–(2.3), which is similar to the global quasi-geostrophic Equations (1.11)–(1.14).

From the second equation in (1.7), we have

$$\operatorname{div} v^1 = g^1 = g^1(\theta, \varphi, \eta, t), \tag{2.4}$$

where

$$g^1 = \frac{\partial}{\partial \eta}\left(K_2\left(L_2 T^0 + \alpha \frac{\partial T^0}{\partial t} - Q\right)\right). \tag{2.5}$$

Now let $\omega^1 = \omega^1(\theta, \varphi, \eta, t)$ be such that

$$\operatorname{div} \omega^1 = g^1. \tag{2.6}$$

We set $\tilde{v}^1 = v^1 - \omega^1$, so that div $\tilde{v}^1 = 0$. Moreover, from the first equation in (1.7), we have

$$fk \times v^1 + \nabla \Phi^1 = h^1 = h^1(\theta, \varphi, \eta, t), \tag{2.7}$$

where

$$h^1 = -L_1 v^0 - \frac{\partial v^0}{\partial t} - \nabla_{v^0} v^0. \tag{2.8}$$

Let $\Psi^1 = \Psi^1(\theta, \varphi, \eta, t)$ be such that

$$fk \times \omega^1 + \nabla \Psi^1 = h^1. \tag{2.9}$$

Then $\widetilde{\Phi}^1 = \Phi^1 - \Psi^1$ and $\widetilde{v}^1 = v^1 - \omega^1$ satisfy

$$fk \times \widetilde{v}^1 + \nabla \widetilde{\Phi}^1 = 0, \qquad \operatorname{div} \widetilde{v}^1 = 0, \tag{2.10}$$

which gives (exactly as in [10]),

$$\widetilde{v}_\theta^1 = 0, \qquad \widetilde{v}_\varphi^1 = \frac{1}{f}\frac{\partial \widetilde{\Phi}^1}{\partial \theta}, \qquad \widetilde{\Phi}^1 \text{ is independent of } \varphi. \tag{2.11}$$

Let $\widetilde{T}^1 = T^1 - T_*^1$, where $T_*^1 = K_2(\partial \Psi^1/\partial \eta)$. Then, from (1.9) we have $\widetilde{T}^1 = K_2(\partial \widetilde{\Phi}^1/\partial \eta)$. Finally, we obtain

$$\widetilde{v}_\theta^1 = 0, \qquad \widetilde{v}_\varphi^1 = \frac{1}{f}\frac{\partial \widetilde{\Phi}^1}{\partial \theta}, \qquad \widetilde{T}^1 = K_2 \frac{\partial \widetilde{\Phi}^1}{\partial \eta}, \qquad \widetilde{\Phi}^1 \text{ is independent of } \varphi. \tag{2.12}$$

By using the change of variables

$$\widetilde{v}^1 = v^1 - \omega^1, \qquad \widetilde{T}^1 = T^1 - T_*^1, \qquad \widetilde{\Phi}^1 = \Phi^1 - \Psi^1 \tag{2.13}$$

and dropping all $\widetilde{}$, we can rewrite (2.1)–(2.3) in the form (2.14)–(2.16) as follows:

$$\left.\begin{aligned} &\frac{\partial v^1}{\partial t} + \nabla_{v^1} v^0 + \nabla_{v^0} v^1 + fk \times v^2 + \nabla \Phi^2 + L_1 v^1 = P^1, \\ &\alpha \frac{\partial T^1}{\partial t} + \alpha \nabla_{v^0} T^1 + \alpha \nabla_{v^1} T^0 + L_2 T^1 - \frac{W(v^2)}{K_2} = R^1, \end{aligned}\right\} \tag{2.14}$$

$$v_\theta^1 = 0, \qquad v_\varphi^1 = \frac{1}{f}\frac{\partial \Phi^1}{\partial \theta}, \qquad T^1 = K_2 \frac{\partial \Phi^1}{\partial \eta}, \qquad \Phi^1 \text{ is independent of } \varphi, \tag{2.15}$$

$$\operatorname{div} \int_0^1 v^2 \, \mathrm{d}\eta = 0, \qquad T^2 = K_2 \frac{\partial \Phi^2}{\partial \eta}, \qquad \Phi^2 = \Phi_S^2 + \mathcal{M}(T^2/K_2). \tag{2.16}$$

We supplement (2.14)–(2.16) with the following initial and boundary-value conditions:

$$\left.\begin{aligned} &\frac{\partial v^1}{\partial \eta} = \gamma_s(v^1 - \widetilde{v}_s^1), \qquad \frac{\partial T^1}{\partial \eta} = \alpha_s(T^1 - \widetilde{T}_s^1), \quad \text{at } \eta = 0, \\ &\frac{\partial v^1}{\partial \eta} = \widetilde{v}_b^1, \qquad \frac{\partial T^1}{\partial \eta} = \widetilde{T}_b^1, \quad \text{at } \eta = 1; \qquad (v^1, T^1)|_{t=0} = (\widetilde{v}_0^1, \widetilde{T}_0^1). \end{aligned}\right\} \tag{2.17}$$

Here $\widetilde{v}_s^1, \widetilde{T}_s^1, \widetilde{v}_b^1, \widetilde{T}_b^1, \widetilde{v}_0^1, \widetilde{T}_0^1$, are given and defined from (2.3) and (2.13) by:

$$\widetilde{v}_s^1 = v_s^1 + \frac{1}{\gamma_s}\frac{\partial \omega^1}{\partial \eta} - \omega^1, \qquad \widetilde{T}_s^1 = T_s^1 + \frac{1}{\alpha_s}\frac{\partial T_*^1}{\partial \eta} - T_*^1,$$

$$\widetilde{v}_b^1 = -\frac{\partial \omega^1}{\partial \eta}, \qquad \widetilde{T}_b^1 = -\frac{\partial T_*^1}{\partial \eta}, \qquad \widetilde{v}_0^1 = v_0^1 - \omega^1, \qquad \widetilde{T}_0^1 = T_0^1 - T_*^1.$$

In (2.14), P^1 and R^1 are defined by:

$$P^1 = -\frac{\partial \omega^1}{\partial t} - \nabla_{w^1} v^0 - \nabla_{v^0}\omega^1 + W(\omega^1)\frac{\partial v^0}{\partial \eta} - L_1\omega^1,$$
$$R^1 = -\alpha\frac{\partial T_*^1}{\partial t} - \alpha\nabla_{v^0}T_*^1 - \alpha\nabla_{\omega^1}T^0 + \alpha W(\omega^1)\frac{\partial T^0}{\partial \eta} - L_2 T_*^1.$$

Moreover, we have:

$$\nabla_{v^1}v^0 = -v_\varphi^1 v_\varphi^0 \cot\theta e_\theta, \qquad \nabla_{v^0}v^1 = -v_\varphi^0 v_\varphi^1 \cot\theta e_\theta, \qquad \nabla_{v^0}T^1 = \nabla_{v^1}T^0 = 0.$$

Then (2.14) can be rewritten as follows:

$$\left.\begin{aligned} &\frac{\partial \Phi^2}{\partial \theta} - 2v_\varphi^1 v_\varphi^0 \cot\theta - f v_\varphi^2 = P_\theta^1, \qquad \alpha\frac{\partial T^1}{\partial t} - \frac{W(v^2)}{K_2} + L_2 T^1 = R^1,\\ &\frac{\partial v_\varphi^1}{\partial t} + f v_\theta^2 + \frac{1}{\sin\theta}\frac{\partial \Phi^2}{\partial \varphi} + L_1 v_\varphi^1 = P_\varphi^1. \end{aligned}\right\} \tag{2.18}$$

As in [10], let us integrate (2.18) with respect to the longitude variable φ over the interval $(0, 2\pi)$; we obtain

$$\left.\begin{aligned} &\frac{\partial v_\varphi^1}{\partial t} + \mathcal{L}_1 v_\varphi^1 + f\overline{v}_\theta^2 = \overline{P}_\varphi^1,\\ &\alpha\frac{\partial T^1}{\partial t} + \mathcal{L}_2 T^1 + \frac{1}{K_2}\int_\eta^1 \frac{1}{\sin\theta}\frac{\partial}{\partial \theta}(\overline{v}_\theta^2 \sin\theta)\,\mathrm{d}\eta' = \overline{R}^1, \end{aligned}\right\} \tag{2.19}$$

$$v_\varphi^1 = \frac{1}{f}\frac{\partial \Phi^1}{\partial \theta}, \qquad v_\theta^1 = 0, \qquad T^1 = K_2\frac{\partial \Phi^1}{\partial \eta}, \qquad \Phi^1 \text{ is independent of } \varphi, \tag{2.20}$$

$$\int_0^1 \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}(\overline{v}_\theta^2 \sin\theta)\,\mathrm{d}\eta' = 0, \tag{2.21}$$

where

$$\overline{v}_\theta^2 = \frac{1}{2\pi}\int_0^{2\pi} v_\theta^2\,\mathrm{d}\varphi, \qquad \overline{R} = \frac{1}{2\pi}\int_0^{2\pi} R\,\mathrm{d}\varphi, \qquad \overline{P}_\varphi^1 = \frac{1}{2\pi}\int_0^{2\pi} P_\varphi^1\,\mathrm{d}\varphi.$$

The initial and boundary conditions read:

$$\left.\begin{aligned} &\frac{\partial v^1}{\partial \eta} = \gamma_s(v^1 - \overline{v}_s^1), \qquad \frac{\partial T^1}{\partial \eta} = \alpha_s(T^1 - \overline{T}_s^1), \quad \text{at } \eta = 0,\\ &\frac{\partial v^1}{\partial \eta} = \overline{v}_b^1, \qquad \frac{\partial T^1}{\partial \eta} = \overline{T}_b^1, \quad \text{at } \eta = 1; \qquad (v^1, T^1)|_{t=0} = (\overline{v}_0^1, \overline{T}_0^1), \end{aligned}\right\} \tag{2.22}$$

where $\overline{v}_s^1, \overline{T}_s^1, \overline{v}_b^1, \overline{T}_b^1, \overline{v}_0^1, \overline{T}_0^1$, are obtained by integration of (2.17) with respect to φ, that is:

$$\overline{v}_s^1 = \frac{1}{2\pi}\int_0^{2\pi} \widetilde{v}_s^1\,\mathrm{d}\varphi$$

and so on.

REMARK 2.1. At the level $O(\varepsilon^2)$, the solution of the second-order approximate Equations (2.1)–(2.3) is given by $(v^1 + \omega^1, T^1 + T^1_*)$. □

We can prove the existence and uniqueness of solutions of (2.19)–(2.22) using the mathematical framework developed in [10], with the assumptions that the solutions (v^0_φ, T^0) of the global quasi-geostrophic Equations (1.11)–(1.14) are sufficiently regular.

3. $(n+1)$th-order approximate equations

In this section, we generalize the results obtained in the previous section to the general $(n+1)$th-order approximate equations, where n is any integer. The approximate equations obtained are important from a numerical point of view, because they provide a simple way to compute the components (v^k, T^k) in the asymptotic expansions (1.3).

At the level $O(\varepsilon^{n+1})$ we obtain:

$$\left.\begin{aligned}
&\varepsilon^{n+1}\left[\frac{\partial v^n}{\partial t} + \sum_{k=0}^{n}\left(\nabla_{v^{n-k}} v^k - W(v^{n-k})\frac{\partial v^k}{\partial \eta}\right) + fk \times v^{n+1} + \nabla\Phi^{n+1} + L_1 v^n\right] \\
&+\varepsilon^{n}\left[\frac{\partial v^{n-1}}{\partial t} + \sum_{k=0}^{n-1}\left(\nabla_{v^{n-1-k}} v^k - W(v^{n-1-k})\frac{\partial v^k}{\partial \eta}\right) + fk \times v^{n} + \nabla\Phi^{n} + L_1 v^{n-1}\right] \\
&\vdots \qquad\qquad\qquad\qquad \vdots \qquad\qquad\qquad\qquad \vdots \\
&\vdots \qquad\qquad\qquad\qquad \vdots \qquad\qquad\qquad\qquad \vdots \\
&+\varepsilon\left[\frac{\partial v^0}{\partial t} + \nabla_{v^0} v^0 + fk \times v^1 + \nabla\Phi^1 + L_1 v^0\right] + [fk \times v^0 + \nabla\Phi^0] = O(\varepsilon^{n+2}).
\end{aligned}\right\} \tag{3.1}$$

$$\left.\begin{aligned}
&\varepsilon^{n+1}\left[\alpha\frac{\partial T^n}{\partial t} + \alpha\sum_{k=0}^{n}\left(\nabla_{v^{n-k}} T^k - W(v^{n-k})\frac{\partial T^k}{\partial \eta}\right) - \frac{W(v^{n+1})}{K_2} + L_2 T^n\right] \\
&+\varepsilon^{n}\left[\alpha\frac{\partial T^{n-1}}{\partial t} + \alpha\sum_{k=0}^{n-1}\left(\nabla_{v^{n-1-k}} T^k - W(v^{n-1-k})\frac{\partial T^k}{\partial \eta}\right) - \frac{W(v^{n})}{K_2} + L_2 T^{n-1}\right] \\
&\vdots \qquad\qquad\qquad\qquad \vdots \qquad\qquad\qquad\qquad \vdots \\
&\vdots \qquad\qquad\qquad\qquad \vdots \qquad\qquad\qquad\qquad \vdots \\
&+\varepsilon\left[\alpha\frac{\partial T^0}{\partial t} + \alpha\nabla_{v^0} T^0 + \alpha\nabla_{v^0} T^0 - \alpha W(v^0)\frac{\partial T^0}{\partial \eta} - \frac{W(v^1)}{K_2} + L_2 T^0\right] - \frac{W(v^0)}{K_2} \\
&\quad = \varepsilon Q + O(\varepsilon^{n+2}),
\end{aligned}\right\} \tag{3.2}$$

$$\left.\begin{aligned} &\operatorname{div}\int_0^1 (v^0+\varepsilon v^1+\varepsilon^2 v^2+\cdots \varepsilon^{n+1}v^{n+1})\,\mathrm{d}\eta = O(\varepsilon^{n+2}),\\ &T^{n+1}=K_2\frac{\partial \Phi^{n+1}}{\partial \eta}, \qquad \Phi^{n+1}=\Phi_S^{n+1}+\mathcal{M}(T^{n+1}/K_2). \end{aligned}\right\} \tag{3.3}$$

From the previous order approximate equations (at the orders up to n), we obtain the following $(n+1)$th-order approximate Equations (3.4)–(3.6):

$$\left.\begin{aligned} &\frac{\partial v^n}{\partial t}+\sum_{k=0}^{n}\left(\nabla_{v^{n-k}}v^k - W(v^{n-k})\frac{\partial v^k}{\partial \eta}\right)+fk\times v^{n+1}+\nabla\Phi^{n+1}+L_1 v^n=0,\\ &\alpha\frac{\partial T^n}{\partial t}+\alpha\sum_{k=0}^{n}\left(\nabla_{v^{n-k}}T^k - W(v^{n-k})\frac{\partial T^k}{\partial \eta}\right)-\frac{W(v^{n+1})}{K_2}+L_2T^n=0, \end{aligned}\right\} \tag{3.4}$$

$$\operatorname{div}\int_0^1 v^n\,\mathrm{d}\eta=0, \qquad T^n=K_2\frac{\partial\Phi^n}{\partial\eta}, \qquad \Phi^n=\Phi_S^n+\mathcal{M}(T^n/K_2), \tag{3.5}$$

$$\operatorname{div}\int_0^1 v^{n+1}\,\mathrm{d}\eta=0, \quad T^{n+1}=K_2\frac{\partial\Phi^{n+1}}{\partial\eta}, \quad \Phi^{n+1}=\Phi_S^{n+1}+\mathcal{M}(T^{n+1}/K_2), \tag{3.6}$$

with the initial and boundary value conditions

$$\left.\begin{aligned} &\frac{\partial v^n}{\partial\eta}=\gamma_s(v^n-v_s^n), \qquad \frac{\partial T^n}{\partial \eta}=\alpha_s(T^n-T_s^n), \quad \text{at } \eta=0,\\ &\frac{\partial v^n}{\partial\eta}=0, \qquad \frac{\partial T^n}{\partial\eta}=0, \quad \text{at } \eta=1; \qquad (v^n,T^n)=(v_0^n,T_0^n), \quad \text{at } t=0. \end{aligned}\right\} \tag{3.7}$$

Using the previous approximate equations (approximate equations up to the order n), we can rewrite (3.4) into the form:

$$\left.\begin{aligned} &\frac{\partial v^n}{\partial t}+\nabla_{v^n}v^0+\nabla_{v^0}v^n-W(v^n)\frac{\partial v^0}{\partial\eta}-W(v^0)\frac{\partial v^n}{\partial \eta}\\ &\quad +fk\times v^{n+1}+\nabla\Phi^{n+1}+L_1v^n=A^n,\\ &\alpha\frac{\partial T^n}{\partial t}+\alpha\nabla_{v^n}T^0+\alpha\nabla_{v^0}T^n-W(v^n)\frac{\partial T^0}{\partial\eta}-W(v^0)\\ &\quad\times\frac{\partial T^n}{\partial\eta}-\frac{W(v^{n+1})}{K_2}+L_2T^n=B^n, \end{aligned}\right\} \tag{3.8}$$

where A^n, B^n are known from the previous approximate equations and are defined by:

$$\begin{aligned} A^n &= -\sum_{k=1}^{n-1}\left(\nabla_{v^{n-k}}v^k-W(v^{n-k})\frac{\partial v^k}{\partial\eta}\right),\\ B^n &= -\sum_{k=1}^{n-1}\left(\nabla_{v^{n-k}}T^k-W(v^{n-k})\frac{\partial T^k}{\partial\eta}\right). \end{aligned}$$

Let us also recall that, from the nth approximate equations, we have

$$\left.\begin{aligned}&\frac{\partial v^{n-1}}{\partial t}+\sum_{k=0}^{n-1}\left(\nabla_{v^{n-1-k}}v^k-W(v^{n-1-k})\frac{\partial v^k}{\partial \eta}\right)+fk\times v^n+\nabla\Phi^n+L_1v^{n-1}=0,\\&\alpha\frac{\partial T^{n-1}}{\partial t}+\alpha\sum_{k=1}^{n-1}\left(\nabla_{v^{n-1-k}}T^k-W(v^{n-1-k})\frac{\partial T^k}{\partial \eta}\right)-\frac{W(v^n)}{K_2}+L_2T^{n-1}=0,\end{aligned}\right\}\tag{3.9}$$

which gives

$$fk\times v^n+\nabla\Phi^n=h^n,\qquad \operatorname{div} v_n=g^n.\tag{3.10}$$

Here

$$h^n=-\frac{\partial v^{n-1}}{\partial t}-\sum_{k=0}^{n-1}\left(\nabla_{v^{n-1-k}}v^k-W(v^{n-1-k})\frac{\partial v^k}{\partial \eta}\right)-L_1v^{n-1},\tag{3.11}$$

and

$$g^n=\frac{\partial}{\partial\eta}\left[K_2\left(\alpha\frac{\partial T^{n-1}}{\partial t}+\alpha\sum_{k=0}^{n-1}\left(\nabla_{v^{n-1-k}}T^k-W(v^{n-1-k})\frac{\partial T^k}{\partial \eta}\right)+L^2T^{n-1}\right)\right],\tag{3.12}$$

are known from the previous approximate equations.

Exactly as for the second-order approximate equations, let $\omega^n=\omega^n(\theta,\varphi,\eta,t)$ be such that

$$\operatorname{div}\omega^n=g^n.\tag{3.13}$$

We set $\widetilde{v}^n=v^n-\omega^n$, then div $\widetilde{v}^n=0$. Let $\Psi^n=\Psi^n(\theta,\varphi,\eta,t)$ be such that

$$fk\times\omega^n+\nabla\Psi^n=h^n.\tag{3.14}$$

Then $\widetilde{\Phi}^n=\Phi^n-\Psi^n$, satisfies

$$fk\times\widetilde{v}^n+\nabla\widetilde{\Phi}^n=0,\qquad \operatorname{div}\widetilde{v}^n=0,\tag{3.16}$$

which gives (exactly as in [10])

$$\widetilde{v}^n_\theta=0,\qquad \widetilde{v}^n_\varphi=\frac{1}{f}\frac{\partial\widetilde{\Phi}^n}{\partial\theta},\qquad \widetilde{\Phi}^n \text{ is independent of } \varphi.\tag{3.17}$$

Let $\widetilde{T}^n=T^n-T^n_*$, where $T^n_*=K_2(\partial\Psi^n/\partial\eta)$. Then from (3.5), we have $\widetilde{T}^n=K_2(\partial\widetilde{\Phi}^n/\partial\eta)$.

Finally, we obtain

$$\widetilde{v}^n_\theta=0,\qquad \widetilde{v}^n_\varphi=\frac{1}{f}\frac{\partial\widetilde{\Phi}^n}{\partial\theta},\qquad \widetilde{T}^n=K_2\frac{\partial\widetilde{\Phi}^n}{\partial\eta},\qquad \widetilde{\Phi}^n \text{ is independent of } \varphi.\tag{3.18}$$

By using the change of variables

$$\widetilde{v}^n = v^n - \omega^n, \qquad \widetilde{T}^n = T^n - T_*^n, \qquad \widetilde{\Phi}^n = \Phi^n - \Psi^n, \tag{3.19}$$

and dropping all $\widetilde{}$, we can rewrite the $(n+1)$th-order approximate equations as

$$\left.\begin{aligned} &\frac{\partial v^n}{\partial t} + \nabla_{v^n} v^0 + \nabla_{v^0} v^n + fk \times v^{n+1} + \nabla \Phi^{n+1} + L_1 v^n = P^n, \\ &\alpha \frac{\partial T^n}{\partial t} + \alpha \nabla_{v^n} T^0 + \alpha \nabla v^0 T^n + L_2 T^n - \frac{W(v^{n+1})}{K_2} = R^n, \end{aligned}\right\} \tag{3.20}$$

$$v_\theta^n = 0, \qquad v_\varphi^n = \frac{1}{f}\frac{\partial \Phi^n}{\partial \theta}, \qquad T^n = K_2 \frac{\partial \Phi^n}{\partial \eta}, \qquad \Phi^n \text{ is independent of } \varphi. \tag{3.21}$$

$$\operatorname{div} \int_0^1 v^{n+1}\, \mathrm{d}\eta = 0, \;\; T^{n+1} = K_2 \frac{\partial \Phi^{n+1}}{\partial \eta}, \;\; \Phi^{n+1} = \Phi_S^{n+1} + \mathcal{M}(T^{n+1}/K_2). \tag{3.22}$$

The initial and boundary value conditions are as follows:

$$\left.\begin{aligned} &\frac{\partial v^n}{\partial \eta} = \gamma_s (v^n - \widetilde{v}_s^n), \qquad \frac{\partial T^n}{\partial \eta} = \alpha_s (T^n - \widetilde{T}_s^n), \quad \text{at } \eta = 0, \\ &\frac{\partial v^n}{\partial \eta} = \widetilde{v}_b^n, \qquad \frac{\partial T^n}{\partial \eta} = \widetilde{T}_b^n, \quad \text{at } \eta = 1; \qquad (v^n, T^n)|_{t=0} = (\widetilde{v}_0^n, \widetilde{T}_0^n), \end{aligned}\right\} \tag{3.23}$$

where $\widetilde{v}_s^n, \widetilde{T}_s^n, \widetilde{v}_b^n, \widetilde{T}_b^n, \widetilde{v}_0^n, \widetilde{T}_0^n$, are known and are defined from (3.7) and (3.19) by:

$$\widetilde{v}_s^n = v_s^n + \frac{1}{\gamma_s}\frac{\partial \omega^n}{\partial \eta} - \omega^n, \qquad \widetilde{T}_s^n = T_s^n + \frac{1}{\alpha_s}\frac{\partial T_*^n}{\partial \eta} - T_*^n,$$

$$\widetilde{v}_b^n = -\frac{\partial \omega^n}{\partial \eta}, \qquad \widetilde{T}_b^n = -\frac{\partial T_*^n}{\partial \eta}, \qquad \widetilde{v}_0^n = v_0^n - \omega^n, \qquad \widetilde{T}_0^n = T_0^n - T_*^n.$$

In (3.20), P^n and R^n are defined by:

$$P^n = A^n - \frac{\partial \omega^n}{\partial t} - L_1 \omega^n - \nabla_{\omega^n} v^0 - \nabla_{v^0} \omega^n + W(\omega^n) \frac{\partial v^0}{\partial \eta},$$

$$R^n = B^n - \alpha \frac{\partial T_*^n}{\partial t} - L_2 T_*^n - \alpha \nabla_{\omega^n} T^0 - \alpha \nabla_{v^0} T_*^n + \alpha W(\omega^n) \frac{\partial T^0}{\partial \eta}.$$

Moreover, we have:

$$\nabla_{v^n} v^0 = -v_\varphi^n v_\varphi^0 \cot\theta e_\theta, \qquad \nabla_{v^0} v^n = -v_\varphi^0 v_\varphi^n \cot\theta e_\theta, \qquad \nabla_{v^0} T^n = \nabla_{v^n} T^0 = 0.$$

Then (3.20) can be rewritten as follows:

$$\left.\begin{aligned} &\frac{\partial \Phi^{n+1}}{\partial \theta} - 2 v_\varphi^n v_\varphi^0 \cot\theta - f v_\varphi^{n+1} = P_\theta^n, \\ &\frac{\partial v_\varphi^n}{\partial t} + L_1 v_\varphi^n + f v_\theta^{n+1} + \frac{1}{\sin\theta}\frac{\partial \Phi^{n+1}}{\partial \varphi} = P_\varphi^n, \\ &\alpha \frac{\partial T^n}{\partial t} + L_2 T^n - \frac{W(v^{n+1})}{K_2} = R^n. \end{aligned}\right\} \tag{3.24}$$

As in [10], let us integrate (3.21)–(3.24) with respect to the longitudinal variable φ over the interval $(0, 2\pi)$. We obtain

$$\left.\begin{aligned}&\frac{\partial v_\varphi^n}{\partial t} + f\overline{v}_\theta^{n+1} + \mathcal{L}_1 v_\varphi^n = \overline{P}_\varphi^n,\\ &\alpha\frac{\partial T^n}{\partial t} + \mathcal{L}_2 T^n + \frac{1}{K_2}\int_\eta^1 \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}(\overline{v}_\theta^{n+1}\sin\theta)\,\mathrm{d}\eta' = \overline{R}^n,\end{aligned}\right\} \tag{3.25}$$

$$\widetilde{v}_\varphi^n = \frac{1}{f}\frac{\partial\Phi^n}{\partial\theta}, \qquad v_\theta^n = 0, \qquad T^n = K_2\frac{\partial\Phi^n}{\partial\eta}, \qquad \Phi^n \text{ is independent of } \varphi, \tag{3.26}$$

$$\int_0^1 \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}(\overline{v}_\theta^{n+1}\sin\theta)\,\mathrm{d}\eta' = 0, \tag{3.27}$$

where

$$\overline{v}_\theta^{n+1} = \frac{1}{2\pi}\int_0^{2\pi} v_\theta^{n+1}\,\mathrm{d}\varphi, \qquad \overline{P}_\varphi^n = \frac{1}{2\pi}\int_0^{2\pi} P_\varphi^n\,\mathrm{d}\varphi, \qquad \overline{R}^n = \frac{1}{2\pi}\int_0^{2\pi} R^n\,\mathrm{d}\varphi.$$

The initial and boundary conditions read:

$$\left.\begin{aligned}&\frac{\partial v^n}{\partial\eta} = \gamma_s(v^n - \overline{v}_s^n), \qquad \frac{\partial T^n}{\partial\eta} = \alpha_s(T^n - \overline{T}_s^n), \quad \text{at } \eta = 0,\\ &\frac{\partial v^n}{\partial\eta} = \overline{v}_b^n, \qquad \frac{\partial T^n}{\partial\eta} = \overline{T}_b^n, \quad \text{at } \eta = 1,\\ &(v^n, T^n) = (\overline{v}_0^n, \overline{T}_0^n), \quad \text{at } t = 0,\end{aligned}\right\} \tag{3.28}$$

where $\overline{v}_s^n, \overline{T}_s, \overline{v}_b^n, \overline{T}_b^n, \overline{v}_0^n, \overline{T}_0^n$, are obtained by integration of (2.26) with respect to φ, that is:

$$\overline{v}_s^n = \frac{1}{2\pi}\int_0^{2\pi} \widetilde{v}_s^n\,\mathrm{d}\varphi$$

and so on.

We can prove existence and uniqueness of solutions of (3.25)–(3.28) by using the mathematical framework developed in [10] with the assumptions that the solutions $(v_\varphi^k, T^k), k < n$ of the previous approximate equations are sufficiently regular.

4. Numerical solutions

In this section, we present some numerical solutions of the stationary form of the second-order approximate Equations (2.19)—(2.22) given by the system (4.5)–(4.8) below. We compare these solutions of the stationary form of the global quasi-geostrophic equations given by (4.1)–(4.4):

$$\mathcal{L}_1 v_\varphi^0 + f\overline{v}_\theta^1 = 0, \qquad \mathcal{L}_2 T^0 + \frac{1}{K_2}\int_\eta^1 \frac{1}{\sin\theta}\frac{\partial(\overline{v}_\theta^1\sin\theta)}{\partial\theta}\,\mathrm{d}\eta' = \overline{Q}, \tag{4.1}$$

$$v^0 = v^0_\varphi e_\varphi = \frac{1}{f}\frac{\partial \Phi^0}{\partial \theta} e_\varphi, \qquad T^0 = K_2 \frac{\partial \Phi^0}{\partial \eta}, \qquad \Phi^0 \text{ is independent of } \varphi, \tag{4.2}$$

$$\int_0^1 \frac{1}{\sin\theta}\frac{\partial(\overline{v}^1_\theta \sin\theta)}{\partial\theta}\,\mathrm{d}\eta' = 0, \tag{4.3}$$

with the boundary conditions

$$\left.\begin{aligned} &\frac{\partial v^0}{\partial \eta} = \gamma_s(v^0 - \overline{v}^0_s), \qquad \frac{\partial T^0}{\partial \eta} = \alpha_s(T^0 - \overline{T}^0_s), \quad \text{at } \eta = 1,\\ &\frac{\partial v^0}{\partial \eta} = 0, \qquad \frac{\partial T^0}{\partial \eta} = 0, \quad \text{at } \eta = 1. \end{aligned}\right\} \tag{4.4}$$

The stationary case of the second-order approximate Equations (2.23)–(2.26) is given by

$$\mathcal{L}_1 v^1_\varphi + f\overline{v}^2_\theta = \overline{P}^1_\varphi, \qquad \mathcal{L}_2 T^1 + \frac{1}{K_2}\int_\eta^1 \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}(\overline{v}^2_\theta \sin\theta)\,\mathrm{d}\eta' = \overline{R}^1, \tag{4.5}$$

$$v^1_\varphi = \frac{1}{f}\frac{\partial \Phi^1}{\partial \theta}, \qquad v^1_\theta = 0, \qquad T^1 = K_2\frac{\partial \Phi^1}{\partial \eta}, \qquad \Phi^1 \text{ is independent of } \varphi, \tag{4.6}$$

$$\int_0^1 \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}(\overline{v}^2_\theta \sin\theta)\,\mathrm{d}\eta' = 0, \tag{4.7}$$

with the boundary conditions

$$\left.\begin{aligned} &\frac{\partial v^1}{\partial \eta} = \gamma_s(v^1 - \overline{v}^1_s), \qquad \frac{\partial T^1}{\partial \eta} = \alpha_s(T^1 - \overline{T}^1_s), \quad \text{at } \eta = 0,\\ &\frac{\partial v^1}{\partial \eta} = \overline{v}^1_b, \qquad \frac{\partial T^1}{\partial \eta} = \overline{T}^1_b, \quad \text{at } \eta = 1. \end{aligned}\right\} \tag{4.8}$$

Let us introduce as in [10] the functions

$$\Pi^0 = \Pi^0(\theta,\eta,t) = \int_\eta^1 \overline{v}^1_\theta(\eta')\,\mathrm{d}\eta', \qquad \Pi^1 = \Pi^1(\theta,\eta,t) = \int_\eta^1 \overline{v}^2_\theta(\eta')\,\mathrm{d}\eta'$$

and the new variable $\zeta = 1 - \eta$. Then, in the new coordinate system (θ,φ,ζ), we can check as in [10] that (4.1)–(4.4) is equivalent to (4.9)–(4.10) given by:

$$\left.\begin{aligned} &L_1 v^0_\varphi + f\frac{\partial \Pi^0}{\partial \xi} = 0, \qquad L_2 T^0 + \frac{1}{K_2 \sin\theta}\frac{\partial}{\partial\theta}(\Pi^0 \sin\theta) = \overline{Q},\\ &\frac{\partial}{\partial \xi}(f v^0_\varphi) + \frac{\partial(T^0/K_2)}{\partial\theta} = 0, \end{aligned}\right\} \tag{4.9}$$

$$\left.\begin{aligned} &\frac{\partial v^0_\varphi}{\partial \xi} = \gamma_s(\overline{v}^0_s - v^0_\varphi), \qquad \frac{\partial T^0}{\partial \xi} = \alpha_s(\overline{T}^0_s - T^0), \quad \text{at } \xi = 1,\\ &\frac{\partial v^0_\varphi}{\partial \xi} = 0, \qquad \frac{\partial T^0}{\partial \xi} = 0, \quad \text{at } \xi = 0. \end{aligned}\right\} \tag{4.10}$$

We can also check that (4.5)–(4.8) is equivalent to (4.11)–(4.12) given by:

$$\left.\begin{aligned} & L_1 v_\varphi^1 + f\frac{\partial \Pi^1}{\partial \xi} = \overline{P}_\varphi^1, \qquad L_2 T^1 + \frac{1}{K_2 \sin\theta}\frac{\partial}{\partial\theta}(\Pi^1 \sin\theta) = \overline{R}^1, \\ & \frac{\partial}{\partial \xi}(f v_\varphi^1) + \frac{\partial (T^1/K_2)}{\partial \theta} = 0, \end{aligned}\right\} \tag{4.11}$$

$$\left.\begin{aligned} & \frac{\partial v_\varphi^1}{\partial \xi} = \gamma_s(\overline{v}_s^1 - v_\varphi^1), \qquad \frac{\partial T^1}{\partial \xi} = \alpha_s(\overline{T}_s^1 - T^1), \quad \text{at } \xi = 1, \\ & \frac{\partial v_\varphi^1}{\partial \xi} = -\overline{v}_b^1, \qquad \frac{\partial T^1}{\partial \xi} = -\overline{T}_b^1, \quad \text{at } \xi = 0, \end{aligned}\right\} \tag{4.12}$$

where

$$\begin{aligned} L_1 v_\varphi &= -\frac{1}{Re_1}\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial v_\varphi}{\partial\theta}\right) - \frac{v_\varphi}{\sin^2\theta}\right] - \frac{1}{Re_2}\frac{\partial}{\partial\xi}\left(K_1\frac{\partial v_\varphi}{\partial\xi}\right), \\ L_2 T &= -\frac{1}{Rt_1 \sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial T}{\partial\theta}\right) - \frac{1}{Rt_2}\frac{\partial}{\partial\xi}\left(K_1\frac{\partial T}{\partial\xi}\right). \end{aligned}$$

To discretize the systems (4.9)–(4.10) and (4.11)–(4.12), we use the latitude-longitude grid with constant interval.

Let $\Delta\theta$ be the latitude grid step, $\Delta\varphi$ the longitude grid step, $\Delta\zeta$ the vertical grid step. Let I, K be two integer numbers; we set

$$\begin{aligned} & \Delta\theta = \frac{\pi}{(I+1)}, \qquad \Delta\zeta = \frac{1}{(K+1)}, \qquad \Delta\varphi = \Delta\theta, \qquad h = (\Delta\theta, \Delta\varphi, \Delta\zeta), \\ & \theta_i = i\Delta\theta, \qquad \zeta_k = k\Delta\zeta, \qquad v_{i,k} = v(\theta_i, \zeta_k), \qquad T_{i,k} = T(\theta_i, \zeta_k). \end{aligned}$$

The Arakawa's C-grid for the global quasi-geostrophic Equations (4.1)–(4.2) is defined as follows:

$$\begin{aligned} &\Pi \quad \text{is defined at the points} \quad (\theta_i, \zeta_k), \\ &v \quad \text{is defined at the points} \quad (\theta_i, \zeta_{k-(1/2)}), \\ &T \quad \text{is defined at the points} \quad (\theta_{i-(1/2)}, \zeta_k). \end{aligned} \tag{4.13}$$

The following figure shows a sample of grid points.

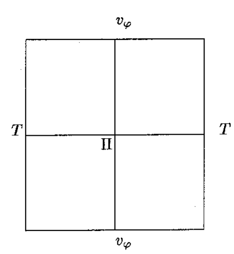

Figure 4.1. Arakawa's C-grid in the (θ, ζ) direction.

The linear operators $L_1, L_2, \partial/\partial\zeta, \partial/\partial\theta$ are approximated by the centered finite-difference scheme of second order at the points $(i, k-\frac{1}{2}), (i-\frac{1}{2}, k), (i,k), (i,k)$, respectively. The corresponding finite-difference operators are denoted by $L_{1,h}, L_{2,h}, \nabla_{\theta,h}, \nabla_{\zeta,h}$, respectively. The linear systems corresponding to (4.9)–(4.10) and (4.11)–(4.12) are solved with Uzawa's conjugate-gradient method. Roughly speaking, we follow these steps:

- Solve the finite dimensional system associated with (4.9)–(4.10) by means of Uzawa's conjugate-gradient method and a change of unknown functions similar to that proposed in [16].
- Compute ω^1 and Ψ^1 which satisfy (2.6) and (2.7), respectively. Then, set $T_*^1 = K_2(\partial\Psi^1/\partial\eta)$.
- Compute the right-hand side $(\overline{P}_\varphi^1, \overline{R}^1)$ of (4.11) and the functions $\overline{v}_s^1, \overline{T}_s^1, \overline{v}_b^1, \overline{T}_b^1$ that appear in the boundary conditions (4.12) and use the formula given in (2.17).
- Solve the finite dimensional system associated with (4.11)–(4.12) by means of Uzawa's conjugate-gradient method and the change of unknown functions similar to that proposed in [16].
- Compute $(v^0 + \varepsilon v^1, T^0 + \varepsilon T^1)$, which is the second-order approximation of (v, T) in the asymptotic expansion (1.3).

For the numerical simulations, we used the following data

$$Q = \overline{Q} = \sigma H(\cos\theta) G(p), \tag{4.14}$$

where

$$H(x) = 1 - 0 \cdot 477(\tfrac{3}{2}x^2 - \tfrac{1}{2}), \qquad G(p) = \frac{\pi}{2} \sin\left[\frac{\pi(P-p)}{(P-p_0)}\right],$$

where σ is a nondimensional constant indicating the intensity of the heating and p is the pressure variable. The heating term Q is chosen according to [17].

The grid sizes and the Reynolds numbers are given by

$$\Delta\theta = \Delta\varphi = \frac{\pi}{61}, \qquad \Delta\zeta = \tfrac{1}{21}, \quad \sigma = 1, \qquad Re_1 = Re_2 = Rt_1 = Rt_2 = 10.$$

Let us recall that in (1.4), the function $K_1(\eta)$ is defined by

$$K_1(\eta) = \left(\frac{p\overline{T}_0}{P\overline{T}}\right), \qquad p = P - (P - p_0)\eta,$$

where $\overline{T}$ satisfies

$$c^2 = R\left(\frac{R\overline{T}}{c_p} - p\frac{\partial\overline{T}}{\partial p}\right) = \text{constant}, \qquad \overline{T}(P) = \frac{P}{\rho R}.$$

Here, P, p_0, R, c_p, ρ and $\overline{T}_0$ are given constants (see [5], [10]).

As noted in [5], $\overline{T}$ can be considered as the climate-average value of the temperature on isobaric surfaces.

For different values of the vertical distribution of the standard temperature $\overline{T}$, the following figures show the latitude-height cross section of the temperature and the zonal wind.

First case. $\overline{T}$ satisfies:

$$100 = R\left(\frac{R\overline{T}}{c_p} - p\frac{\partial \overline{T}}{\partial p}\right).$$

Figures 4.2 and 4.3 show the latitude-height cross section of the temperature for the first and the second-order approximate equations, respectively.

Figures 4.4 and 4.5 show the latitude-height cross section of the zonal wind for the first and the second-order approximate equations, respectively.

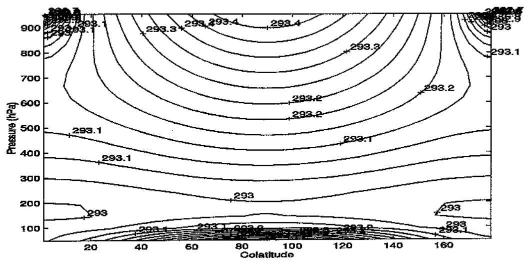

Figure 4.2. Latitude-height cross-section of the zonally averaged temperature: T^0.

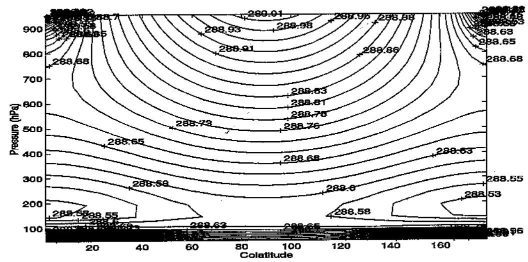

Figure 4.3. Latitude-height cross-section of the zonally averaged temperature: $T^0 + \varepsilon T^1$.

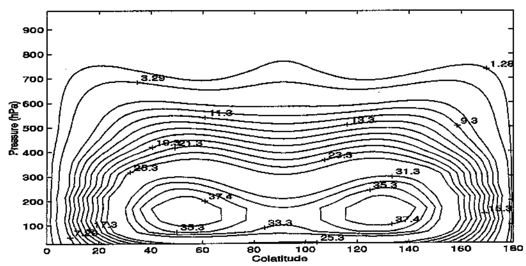

Figure 4.4. Latitude-height cross-section of the zonally averaged zonal wind: v_φ^0.

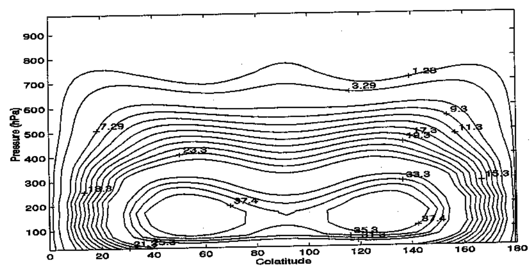

Figure 4.5. Latitude-height cross-section of the zonally averaged zonal wind: $v_\varphi^0 + \varepsilon v_\varphi^1$.

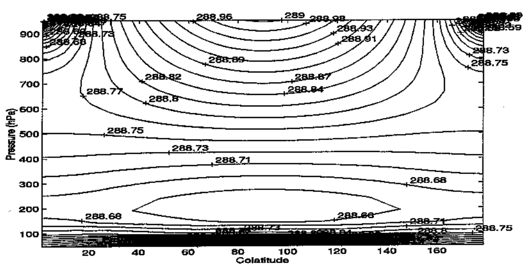

Figure 4.6. Latitude-height cross-section of the zonally averaged temperature: T^0.

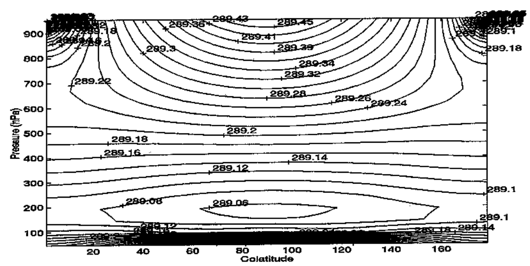

Figure 4.7. Latitude-height cross-section of the zonally averaged temperature: $T^0 + \varepsilon T^1$.

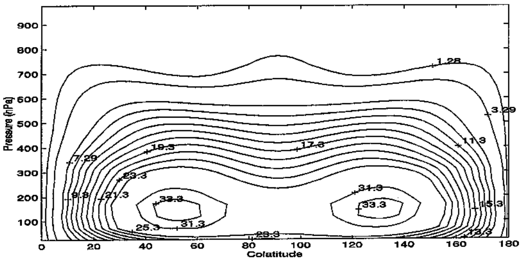

Figure 4.8. Latitude-height cross-section of the zonally averaged zonal wind: v_φ^0.

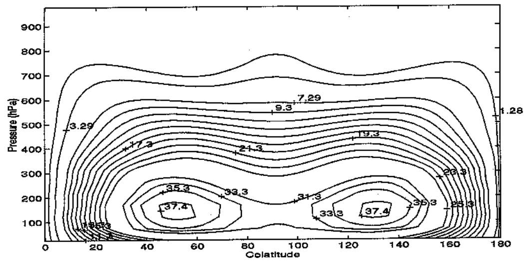

Figure 4.9. Latitude-height cross-section of the zonally averaged zonal wind: $v_\varphi^0 + \varepsilon v_\varphi^1$.

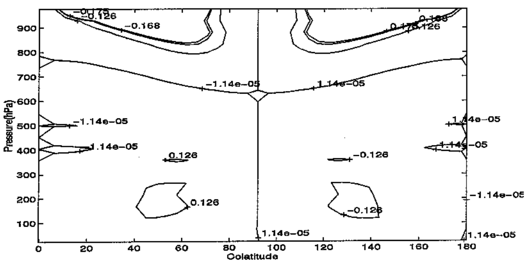

Figure 4.10. Latitude-height cross-section of the zonally averaged meridional wind: εv_θ^1.
First case

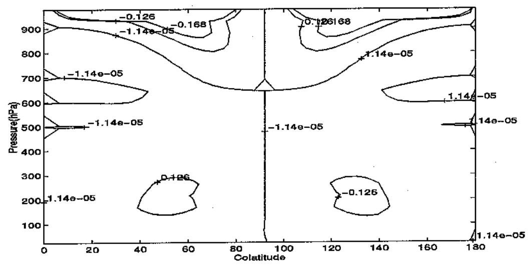

Figure 4.11. Latitude-height cross-section of the zonally averaged meridional wind: εv_θ^1.
Second case.

Second case. $\overline{T}$ satisfies:

$$225 = R\left(\frac{R\overline{T}}{c_p} - p\frac{\partial \overline{T}}{\partial p}\right).$$

Figures 4.6 and 4.7 show the latitude-height cross section of the temperature for the first and the second-order approximate equations, respectively.

Figures 4.8 and 4.9 show the latitude-height cross section of the zonal wind for the first and the second order approximate equations, respectively.

Figures 4.10 and 4.11 show the latitude-height cross section of the zonally averaged meridional wind for the second-order approximate equations.

The results obtained with the first- and the second-order models are comparable in shape and magnitude as far as zonal velocity and temperature are concerned. Moreover, they show for these quantities several of the observed features of the real atmosphere and the CCM2 model (see [4], [16]). In fact, in the lower part of the atmosphere, the general tendency is for the atmosphere to decrease with height. This region, called troposphere, is the portion of the atmosphere in direct contact with the earth's surface. Above the troposphere, the temperature generally stays constant or increases with height. This portion of the atmosphere is called stratosphere. The division between the two regions is the tropopause, [18]. Figures 4.2, 4.3, 4.6 and 4.7 show that the first- and second-order models reproduce the main patterns of the atmosphere. Moreover, the tropopause is located at nearly 200 K, which is in agreement with the real atmosphere as well as the simulations obtained with the CCM2 model ([4], [16]).

The main difference between the simulations of the first- and the second-order approximate equations appears in Figures 4.10 and 4.11, which represent the meridional wind εv_θ^1 obtained with the second-order approximate equations. It should be noted that in the first-order approximation, the meridional component of the velocity is equal to zero (*i.e.* $v_\theta^0 = 0$), whereas, as can be seen in the Figures 4.10 and 4.11, the meridional component of the velocity εv_θ^1 in the second model is sufficiently significant to justify the model: it reaches the magnitude $0{\cdot}126$ in the tropaupose for $\varepsilon = 0{\cdot}02$.

Let us also point out that, when the constant σ is large (*i.e.* the intensity of the heating term is large), let us say 10^3, the term $(\varepsilon v^1, \varepsilon T^1)$ becomes very large.

Conclusions

We had two objectives in this article. The first one was to validate numerically the quasi-geostrophic model derived in [10] by purely mathematical asymptotic analysis. We have been able, with the very simple model of [10] to recover the general features of the atmosphere climate, namely the zonal wind and the temperature, as computed by the much more refined and expensive CCM2 model (a full three-dimensional simulation of the primitive (Navier-Stokes-type) equations).

The second objective was to improve the model in [10] by the introduction of the second- (and higher-) order approximation(s) in the asymptotic expansions with respect to ε. With the second-order approximation we recover again the general features concerning the zonal wind and the temperature, which are valid for both the CCM2 model and the model introduced in [10]. We find also a significant nonzero meridional wind which does not exist in the model in [10].

By considering asymptotic expansions of the primitive equations of the atmosphere with respect to the Rossby number, we derived a series of simple equations for the nth-order approximations of the primitive equations of the atmosphere. We showed that the nth-order approximate equations can be rewritten in a form similar to the global quasi-geostrophic equation derived in [10]. It was shown in [10] that the (first-order) global quasi-geostrophic equations are linear, and that at the first order, wind travels toward east or west. Using the second-order approximation, we recovered the meridional motion, even though it is small in magnitude as expected.

From the computational point of view, since the (first-order) global quasi-geostrophic equations are linear, it is relatively easy to simulate the flow, with special care needed only for the difficulty caused by the singularities at the poles.

The second- and higher-order models are in the same form as the global quasi-geostrophic model with different right-hand sides depending on the approximate solutions of the previous orders. Therefore, it is also relatively easy for us to implement these higher-order models, using the subroutines developed for the first-order approximation. Although we did not verify this, we anticipate that the higher-order approximations should provide better approximations of the real atmospheric flow or, more precisely, the solutions of the primitive equations.

The simulations we obtain resemble in a number of respects the simulations obtained by the NCAR CCM2, which is a well accepted standard in meteorology. Indeed, our models simulate the troposphere, the stratosphere and tropopause. The location of the tropopause is roughly the same as that derived from the CCM2 model [4]. This provides a numerical justification of the Rossby asymptotics we employed in this article. Notice, however, that the magnitudes of our simulated fields are different from the magnitudes of the corresponding fields obtained with the CCM2 model. This main difference may be attributed to the choice of physical parameters (such as the heating term) used in our simulations, which are different from the ones used for the CCM2 simulations.

At this point, we are not able to conclude that $(v^0 + \varepsilon v^1, T^0 + \varepsilon T^1)$ will always give a better approximation of the solution of the primitive Equation (1.1)–(1.2) than (v^0, T^0) is. Moreover, it appears in some cases that $(\varepsilon v^1, \varepsilon T^1)$ is large compared to (v^0, T^0), particularly when the magnitude of the heating term Q increases through the constant σ in (4.14). The physical validity of the GQG model from [10] and of the present higher-order model will be investigated elsewhere.

In conclusion, we have shown that, by taking into account suitable physical aspects of the problem, we can obtain information on a flow without the need of large-scale computations. Of course, this does not preclude the utilization of very large computations when more details are needed or for the validation of simpler models.

Acknowledgments

This work was partially supported by the National Science Foundation under the Grants NSF-DMS9400615, NSF-DMS-960233071, by the Office of Naval Research under the Grants N00014-91-J-1140, N00014-96-1-0-425, and by the Research Fund of Indiana University.

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